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CHARACTERISTIC PROPERTIES

OF THE

SEGMENTED RATIONAL MINMAX APPROXIMATION PROBLEM

Charles L. Lawson

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C. L. Lawson

0. Introduction

A rational minmax approximator for a continuous real valued function on a closed bounded real interval exhibits a characteristic balancing of the extremes of the error curve (Ref. 1, p. 55). This property has been exploited in some of the methods which have been devised for the numerical solution of the rational minmax approximation problem (Ref. 2, 3, 5). Such methods strive iteratively to improve the balance of the extremes of the error curve.

In this paper, we show that there is also a property of balanced maximum errors associated with the segmented rational minmax problem. It is a sufficient but not a necessary condition for a solution. The segmented problem need not have a unique solution, but it always has some solution which has the balanced error property.

Numerical solution methods for the segmented minmax problem can be based upon this property. In a separate paper (Ref. 4), we describe such a method and give some numerical examples.

Section 1 is devoted to three independent lemmas which identify the properties of rational minmax approximation and segmentation upon which the remainder of the development rests. In Section 2, the segmented rational minmax approximation problem is stated, and the existence of a solution is deduced.

In Section 3, the existence of a solution having the balanced error property is established. In Section 4, some inequalities similar to those known in the linear least maximum problem are established, and the balanced error property is shown to be a sufficient condition for a solution.

In Section 5, it is shown that, for any initial position of the breakpoints defining the segmentation, there is a continuous transformation of the breakpoints which permits the maximum error to descend to its minimum value. Section 6 provides some examples illustrating the lack of convexity in this problem.

1. Three basic lemmas.

Let f be a continuous real valued function on the nondegenerate closed bounded real interval $[\alpha,\beta]$. Let n and d be non-negative integers. Let G be the class of rational functions whose numerators and denominators are polynomials of degrees not exceeding n and d respectively. For real numbers v and w satisfying $\alpha \leq v \leq w \leq \beta$, define the minmax error function h by

$$h(v,w) = \min_{g \in G} \max_{v \leq x \leq w} |f(x) - g(x)|$$

Lemma 1. <u>The function</u> h <u>is continuous on the compact region of</u> vw-<u>space defined by</u> $\alpha \leq v \leq w \leq \beta$.

<u>Proof.</u> Let v_o and w_o satisfy $\alpha \le v_o \le w_o \le \beta$. Suppose h is not continuous at (v_o, w_o). Then there exists an $\epsilon > o$ and sequences $\{v_i\}$ and $\{w_i\}$ with $\alpha \le v_i \le w_i \le \beta$, $i = 1, 2, \cdots$, such that $\lim_i v_i = v_o$, $\lim_i w_i = w_o$, and

(1) $\quad |h(v_i, w_i) - h(v_o, w_o)| > \epsilon$ for all $i = 1, 2, \cdots$.

Let g_i denote a member of G which is the minmax approximator for f on $[v_i, w_i]$, $i = 0, 1, \cdots$, normalized so that the coefficient of largest magnitude in g_i is 1. An existence theorem for these minmax approximators is given in (Ref. 1, p. 53).

By definition $|f(x) - g_o(x)|$ is bounded by $h(v_o, w_o)$ for $x \epsilon [v_o, w_o]$. Consequently the continuity of f and g_o permits us to choose a $\delta > 0$ such that $|f(x) - g_o(x)| < h(v_o, w_o) + \epsilon$ for x in the closed interval I between $\max\{v_o - \delta, \alpha\}$ and $\min\{w_o + \delta, \beta\}$. Without loss of generality, we will assume the points v_i and w_i, $i = 1, 2, \cdots$, lie in the closed interval I.

The definitions of h and δ assure that

$$h(v_i, w_i) \le \max_{v_i \le x \le w_i} |f(x) - g_o(x)| \le h(v_o, w_o) + \epsilon$$

Along with inequality (1) this implies

(2) $\quad h(v_i, w_i) < h(v_o, w_o) - \epsilon \qquad i = 1, 2, \cdots$

If $v_o = w_o$, then $h(v_o, w_o) = 0$, in which case (2) is impossible and Lemma 1 is established. We proceed to complete the proof for the case $v_o < w_o$.

The normalization of the rational forms $g_1, g_2, \cdots$, assures that there is a subsequence of $\{g_i\}$ whose corresponding coefficients form convergent sequences. Without loss of generality we will assume the sequence $\{g_i\}$ has this property. Let g^* denote the rational form whose coefficients are respectively the limits of the sequences of corresponding coefficients in the sequence $\{g_i\}$ The normalization assures that the coefficient of largest magnitude in g^* is 1.

Let x' be any point satisfying $v_o < x' < w_o$. Then for all sufficiently large i the point x' also satisfies $v_i < x' < w_i$, whence

(3) $\left| f(x') - g_i(x') \right| \leq h(v_i, w_i) \leq h(v_o, w_o) - \epsilon$ for sufficiently large i. It follows that, unless x' is a zero of its denominator, g^* satisfies $\left| f(x') - g^*(x') \right| \leq h(v_o, w_o) - \epsilon$

Since x' was chosen arbitrarily in (v_o, w_o) this relation holds for all x in (v_o, w_o) except zeros of the denominator of g^*.

Any point x in (v_o, w_o) which is an isolated zero of the denominator of g^* must also be a zero of the numerator of g^* with at least as great multiplicity, for otherwise inequality (3) would be violated in a neighborhood of x for sufficiently large i. Furthermore the denominator of g^* is not identically zero, for then inequality (3) would require that the same be true of the numerator of g^* contradicting the statement that one of the coefficients of g^* is 1. It follows that if g' is the rational form obtained from g^* by removing all polynomial factors common to the numerator and denominator of g^* then g' satisfies

$$|f(x) - g'(x)| \leq h(v_o, w_o) - \epsilon \text{ for all } x \in (v_o, w_o)$$

Under these circumstances g' cannot have a pole at v_o or w_o and thus this bound for $|f(x) - g'(x)|$ is uniform throughout the closed interval $[v_o, w_o]$.

This implies that g' is a better approximator for f on $[v_o, w_o]$ than the best approximator, g_o, whose maximum error is $h(v_o, w_o)$. This contradiction followed from the assumption that h was not continuous at (v_o, w_o). Consequently h must be continuous at (v_o, w_o). This completes the proof of Lemma 1.

Lemma 2. The minmax error function h as defined preceding Lemma 1 is non-increasing in its first variable and non-decreasing in its second variable.

Proof. Let $v_1 \leq v_o \leq w_o \leq w_1$. Let g_1 be the least maximum approximator in G for f on $[v_1, w_1]$.

Then

$$h(v_1, w_1) = \max_{v_1 \leq x \leq w_1} |f(x) - g_1(x)| \geq \max_{v_o \leq x \leq w_o} |f(x) - g_1(x)|$$

$$\geq \min_{g \in G} \max_{v_o \leq x \leq w_o} |f(x) - g(x)| = h(v_o, w_o)$$

Lemma 3. Let m be an integer exceeding 1 and let u_i and v_i be numbers satisfying

$$\alpha = u_o \leq u_1 \leq \cdots \leq u_{m-1} \leq u_m = \beta$$

and

$$\alpha = v_o \leq v_1 \leq \cdots \leq v_{m-1} \leq v_m = \beta$$

Then, unless $u_i = v_i$ for all i, there exist indices j and k such that the following proper inclusions hold:

$$[u_{j-1}, u_j] \subset [v_{j-1}, v_j]$$

and

$$[v_{k-1}, v_k] \subset [u_{k-1}, u_k]$$

Proof. Suppose $u_i \neq v_i$ for some i. Let s be the first index for which inequality holds and without loss of generality assume $u_s < v_s$. Then the lemma is established by letting j = s and letting k be the first index greater than j for which $u_k \geq v_k$.

2. The segmented rational minmax approximation problem.

Let f be a continuous function on $[\alpha, \beta]$ as in Section 1. Let an integer $m \geq 2$ specify the number of contiguous subintervals into which $[\alpha, \beta]$ is to be partitioned by the selection of m-1 breakpoints u_i, i = 1 ..., m-1, satisfying

(4) $$\alpha = u_o \leq u_1 \leq u_2 \leq \cdots \leq u_{m-1} \leq u_m = \beta$$

Let n_i and d_i, i = 1, ..., m, be non-negative integers and let G_i denote the set of rational functions whose numerators and denominators are respectively polynomials of degrees not exceeding n_i and d_i.

For $i = 1, \ldots, m$, and for u_{i-1} and u_i satisfying $\alpha \leq u_{i-1} \leq u_i \leq \beta$, define the minmax error function for the i^{th} subinterval by

$$h_i\,(u_{i-1}, u_i) = \min_{g_i \in G_i} \; \max_{u_{i-1} \leq x \leq u_i} |f(x) - g_i\,(x)|$$

By Lemma 1 each of these functions h_i is continuous on its domain of definition.

Let U denote the subset of $(m+1)$ - space consisting of those vectors $u = (u_o, u_1, \ldots, u_m)$ whose components satisfy (4). On the set U define the maxminmax function μ by

$$\mu\,(u) = \max\ \{h_i\,(u_{i-1}, u_i) : i - 1, \ldots, m\}$$

Our problem is to minimize μ over U.

The continuity of the functions h_i implies the continuity of μ. The existence of a solution vector u^* is then an immediate consequence of the fact that U is compact.

For our later use we introduce the following definitions:

$$\tau = \min\ \{\mu\,(u) : u \in U\}$$

$$U^* = \{u : \mu\,(u) = \tau\}$$

$$\nu\,(u) = \min\ \{h_i\,(u_{i-1}, u_i) : i = 1, \ldots, m\}$$

A vector u will be called <u>balanced</u> if $h_i(u_{i-1}, u_i) = \mu\,(u)$, $i = 1, \ldots, m$.

Note that τ can be called the <u>minmaxminmax</u> error for the problem. It will be shown that $\tau = \max\ \{\nu\,(u) : u \in U\}$ and thus τ also deserves the title of <u>maxminminmax</u> error.

3. Existence of a balanced solution vector.

At this point it will be useful to introduce a closely related dynamic programming problem.

Define:

$$e_1(u_1) = h_1\,(\alpha, u_1)$$

and

$$(5) \qquad e_i(u_i) = \min \quad \max\ \{e_{i-1}(u_{i-1}), h_i(u_{i-1}, u_i)\} \qquad i = 2, \ldots, m$$

To relate this to the problem formulated in Section 2 note that $e_i(u_i)$ is the minmaxminmax error for the i-segment problem on the interval $[\alpha,u_i]$. In particular $e_m(\beta) = \tau$. Equation (5) represents the fact that an i-segment minmaxminmax approximator on $[\alpha,u_i]$ may be found by searching on the single variable u_{i-1} for the most favorable combination of an (i-1)-segment minmaxminmax approximator on $[\alpha,u_{i-1}]$ and a one-segment minmax approximator on $[u_{i-1},u_i]$.

<u>Theorem 1. Statements A, C, and D are valid for</u> $1 \leq i \leq m$ <u>and statement B for</u> $2 \leq i \leq m$.

<u>A.</u> $e_i(\alpha) = 0$

<u>B.</u> <u>Given</u> $v_i \in [\alpha,\beta]$, <u>there exists</u> $v_{i-1} \in [\alpha,v_i]$ <u>such that</u> $e_i(v_i) = e_{i-1}(v_{i-1}) = h_i(v_{i-1},v_i)$

<u>C.</u> e_i <u>is non-decreasing on</u> $[\alpha,\beta]$

<u>D.</u> e_i <u>is continuous on</u> $[\alpha,\beta]$

An immediate consequence of statement B in the above theorem is the following theorem which, for the case $i = m$, asserts the existence of a balanced solution vector.

<u>Theorem 2. Given</u> $v_i \in [\alpha,\beta]$, <u>there exist</u> v_j, $j = 1, \ldots, i-1$, <u>such that</u> $\alpha = v_0 \leq v_1 \leq \cdots \leq v_i$ <u>and</u> $e_i(v_i) = h_j(v_{j-1},v_j)$, $j = 1, \ldots, i$.

<u>Proof of Theorem 1.</u>

Statement A, for $i = 1, \ldots, m$, follows directly from the fact that $h_i(\alpha,\alpha) = 0$, $i = 1, \ldots, m$.

Statements C and D are valid for $i = 1$ due to Lemmas 2 and 1 respectively.

We will now prove B, C, and D for $i > 1$ under the induction hypothesis that C and D are valid for $i - 1$.

To prove B let $v_i \in [\alpha,\beta]$ be given. On the interval $\alpha \leq u_{i-1} \leq v_i$, the function $e_{i-1}(u_{i-1})$ is non-decreasing and vanishes at the left end, whereas $h_i(u_{i-1},v_i)$, considered as a function of u_{i-1} only, is non-increasing and vanishes at the right end. Since both e_{i-1} and h_i are continuous, there must be a point v_{i-1} (not necessarily unique) in $[\alpha,v_i]$ at which $e_{i-1}(v_{i-1}) = h_i(v_{i-1},v_i)$. Such a point obviously provides the minimum value, among $u_{i-1} \in [\alpha,v_i]$, of $\max\left\{e_{i-1}(u_{i-1}),\ h_i(u_{i-1},v_i)\right\}$. This establishes statement B.

For later use we note that the point v_{i-1} also maximizes the value of $\min\{e_{i-1}(u_{i-1}),\ h_i(u_{i-1},v_i)\}$ among $u_{i-1} \in [\alpha,v_i]$ This permits an alternative definition of e_i for $i > 1$:

$$(6)\quad e_i(u_i) = \max_{\alpha \leq u_{i-1} \leq u_i} \min\{e_{i-1}(u_{i-1}),\ h_i(u_{i-1},u_i)\}$$

To prove C let $v_i \leq w_i$ be given. Using B, there exists $v_{i-1} \in [\alpha,v_i]$ such that $e_i(v_i) = e_{i-1}(v_{i-1}) = h_i(v_{i-1},v_i)$. Then for $u_{i-1} \in [\alpha,v_{i-1}]$

$$\begin{aligned} e_i(v_i) = h_i(v_{i-1},v_i) &\leq h_i(u_{i-1},v_i) \leq h_i(u_{i-1},w_i) \\ &\leq \max\{e_{i-1}(u_{i-1}),\ h_i(u_{i-1},w_i)\} \end{aligned}$$

while for $u_{i-1} \in [v_{i-1},w_i]$

$$e_i(v_i) = e_{i-1}(v_{i-1}) \leq e_{i-1}(u_{i-1}) \leq \max\{e_{i-1}(u_{i-1}),\ h_i(u_{i-1},w_i)\}$$

Thus for all $u_{i-1} \in [\alpha,w_i]$

$$e_i(v_i) \leq \max\{e_{i-1}(u_{i-1}),\ h_i(u_{i-1},w_i)\}$$

and so $e_i(v_i) \leq e_i(w_i)$ which establishes C.

For the proof of D let $v_i \in [\alpha,\beta]$ and $\varepsilon > 0$.
Using the motonicity established in C it will suffice to prove the existence of a $\delta > 0$ such that, if $v_i \neq \beta$, $v_i < u_i < v_i + \delta$ implies $e_i(u_i) \leq e_i(v_i) + \varepsilon$

and, if $v_i \neq \alpha$,

$$v_i \geq u_i \geq v_i - \delta \text{ implies } e_i\ (u_i) \geq e_i\ (v_i) - \epsilon$$

By statement B there exists $v_{i-1}\ \epsilon[\alpha,\ v_i]$ such that $e_i(v_i) = e_{i-1}(v_{i-1}) = h_i(v_{i-1},\ v_i)$. By the continuity of h_i (Lemma 1), if $\delta > 0$ is sufficiently small, then $|h_i(v_{i-1},\ u_i) - h_i\ (v_{i-1},\ v_i)| < \epsilon$

For $v_i \neq \beta$ we may assume δ is smaller than $\beta - v_i$. Then for $v_i < u_i < v_i + \delta$

$$e_i(u_i) \leq \max\ \{e_{i-1}(v_{i-1}),\ h_i(v_{i-1},\ u_i)\}$$

$$= h_i(v_{i-1},\ u_i) \leq h_i(v_{i-1},v_i) + \epsilon = e_i(v_i) + \epsilon$$

Similarly for $v_i \neq \alpha$, we may assume $\delta < v_i - \alpha$. Then we wish to consider u_i satisfying $v_i < u_i < v_i - \delta$, but two cases arise, depending on whether $v_{i-1} = v_i$ or $v_{i-1} < v_i$. In the former case we have $e_i(v_i) = h_i(v_{i-1},v_i) = 0$ and thus, by C, $e_i(u_i) = 0$ for all $u_i\ \epsilon[\alpha,v_i]$. Thus we certainly have $e_i(u_i) \geq e_i(v_i) - \epsilon$.

In the latter case we may assume δ is smaller than $v_i - v_{i-1}$ so that v_{i-1} lies in $[\alpha,u_i]$. Then using the alternative definition (6) for e_i we obtain

$$e_i(u_i) \geq \min\ \{e_{i-1}(v_{i-1}),\ h_i(v_{i-1},u_i)\}$$

$$= h_i(v_{i-1},u_i) \geq h_i(v_{i-1},v_i) - \epsilon$$

This completes the proof of statement D and hence of Theorem 1.

4. Sufficiency of the balanced error property

In the previous section the existence of a balanced solution vector was established. In this section it will be shown that every balanced vector is a solution vector. We continue to use the notation introduced in Section 2.

Lemma 4. For any u and v in U

$$\nu(u) \equiv \min_i h_i(u_{i-1},u_i) \leq \max_i h_i(v_{i-1},v_i) \equiv \mu(v)$$

Proof. If $u = v$, the lemma is trivially true. Suppose $u \neq v$. Then by Lemma 3 there is an index j such that $[u_{j-1},u_j] \subset [v_{j-1},v_j]$. Therefore,

$$\nu(u) \leq h_j(u_{j-1},u_j) \leq h_j(v_{j-1},v_j) \leq \mu(v)$$

where the center inequality is due to Lemma 2.

Lemma 5. For any $u \in U$, $\nu(u) \leq \tau \leq \mu(u)$.

Proof. The second inequality follows directly from the definition of τ. To establish the first inequality let u^* be a solution vector and apply Lemma 4, obtaining $\nu(u) \leq \mu(u^*) = \tau$.

An immediate consequence of Lemma 5 is

Lemma 6. A balanced vector is a solution vector.

5. Possibility of descent to a solution

Theorem 3. For any $v \in U$ and $v^* \in U^*$ there is a piecewise linear path P in U connecting v to v^* and having the property that $\mu(u)$ is non-increasing as u moves along P from v to v^*.

Proof. A vector valued function u will be defined which maps a real interval $0 \leq t \leq k$ (for some $k < m$) onto a subset P of U in such a way that P has the properties stated in Theorem 3.

1. Set $t_c = 0$. Set $u_i(0) = v_i, i = 0, 1, \ldots, m$
2. If $u_i(t_c) = v_i^*$, $i = 0, \ldots, m$, then set $k = t_c$ and quit, otherwise go to step 3.
3. Let j be the first index such that $[u_{j-1}(t_c), u_j(t_c)]$ is a proper subset of $[v^*_{j-1},v^*_j]$.

We will call $[u_{j-1}(t),u_j(t)]$ the key variable interval for the current iteration and $[v^*_{j-1},v^*_j]$ the key target interval. The effect of steps 4a and 5a will be to expand the key variable interval so that it coincides with the key target interval when t reaches t_c+1. Steps 4b and 5b provide for components in the path of the expansion to be carried along rather than being bypassed.

4.a. For $t_c < t < t_c + 1$, let $u_{j-1}(t)$ vary linearly taking the value v^*_{j-1} at $t_c + 1$.

4.b. If an index $i < j-1$ satisfies $u_i(t_c) \in [v^*_{j-1}, u_{j-1}(t_c)]$, then, since the value $u_{j-1}(t)$ varies from the right to the left end of this interval as t varies from t_c to $t_c + 1$, there must be a point t_i at which $u_{j-1}(t_i) = u_i(t_c)$. Define $u_i(t)$ to be constant for $t_c \leq t \leq t_i$ and to be equal to $u_{j-1}(t)$ for $t_i \leq t \leq t_c + 1$.

5.a. For $t_c < t < t_c + 1$, let $u_j(t)$ vary linearly taking the value v^*_j at $t_c + 1$.

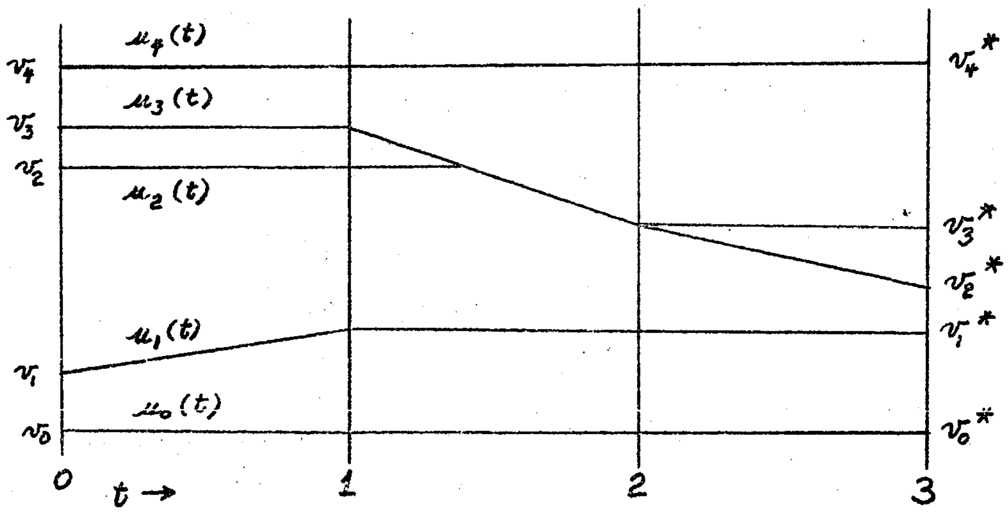
5.b. If an index $i > j$ satisfies $u_i(t_c) \in [u_j(t_c), v^*_j]$ then there must be a point t_i at which $u_j(t_i) = u_i(t_c)$. Define $u_i(t)$ to be constant for $t_c \leq t \leq t_i$ and to be equal to $u_j(t)$ for $t_i \leq t \leq t_c + 1$.

6. For each index i not treated in steps 4 or 5, define $u_i(t)$ to be constant for $t_c < t < t_c + 1$.

7. Replace t_c by $t_c + 1$ and return to step 2.

<u>Remark 1</u>

The following graph illustrates a set of functions $u_i(t)$ defined by this construction:



In this illustration the key interval for the first iteration, i.e., as t varies from 0 to 1, is $[u_0(t), u_1(t)]$. The succeeding key intervals are $[u_3(t), u_4(t)]$ and $[u_2(t), u_3(t)]$. Note that $u_2(t)$ coincides with $u_3(t)$ in the latter part of the interval $1 \leq t \leq 2$.

Remark 2. The existence of the index j needed in step 3 is assured by Lemma 3.

Remark 3. In steps 4a and 5a at least one of the components u_{j-1} or u_j is not equal to its final value (v^*_{j-1} or v^*_j respectively) at $t=t_c$ but both are equal to their final values at $t = t_c + 1$. Both of these two components remain constant for $t \geq t_c + 1$. Thus at least one previously unstabilized component stabilizes on each iteration and so the procedure terminates after at most m-1 iterations.

Remark 4. The function $\mu(u)$ can increase with increasing t only if one of the functions $h_i(u_{i-1}, u_i)$ increases as t increases. For $h_i(u_{i-1}, u_i)$ to increase it is necessary that either u_{i-1} decrease or u_i increase and that u_{i-1} be distinct from u_i.

In the given construction only the key variable interval is permitted to move in this manner. Since it is covered by its target interval during the move it follows from Lemma 2 that $h_j(u_{j-1}, u_j) \leq h_j(v^*_{j-1}, v^*_j) \leq \tau$. This move cannot cause an increase in μ because μ is never less than τ. Thus μ is non-increasing as t goes from 0 to k, i.e. as u goes from v to v^* in the prescribed manner.

The set $\{u(t): 0 \leq t \leq k\}$ has, therefore, all of the properties required of the path P. This concludes the proof of Theorem 3.

6. Counterexamples

In the light of the favorable descent properties stated in Theorem 3 it is natural to inquire whether μ is a convex function on U. Example 1 below shows that this need not be the case. The second example shows that the solution set U^* can fail to be convex although Theorem 3 does imply that U^* is arcwise connected.

Example 1.

This example shows that the function μ can fail to be convex and that μ can have weak local minima which are not global minima. Define f (x) for $|x| \leq 7$ by linear interpolation in the following table:

x	-7	-5	-3	-1	1	3	5	7
f (x)	-3	-3	-1	-1	1	1	3	3

Consider the problem of approximating f by two constant functions, i.e. $m = 2$, $n_1 = n_2 = d_1 = d_2 = 0$. The space U of possible break point vectors consists of all vectors of the form $(-7, u_1, 7)$ with $-7 \leq u_1 \leq 7$. The functions h_1, h_2, and μ are then given by linear interpolation in the following table:

u_1	-7	-5	-3	-1	0	1	3	5	7
$h_1(-7,u_1)$	0	0	1	1	1.5	2	2	3	3
$h_2(u_1, 7)$	3	3	2	2	1.5	1	1	0	0
$\mu(-7,u_1,7)$	3	3	2	2	1.5	2	2	3	3

The function μ is seen to be non-convex. If f is redefined in the interval $[-1,1]$ to be $\sin (x\,\pi/2)$ then the new μ will be non-convex in every neighborhood of the solution vector $(-7, 0, 7)$.

The function μ has many weak local minima, for example, every point in the open interval between $u_1 = -3$ and $u_1 = -1$. The only strict minimum is at $u_1 = 0$.

Example 2.

This example shows that U^* can fail to be convex. Define f (x) for $|x| \leq 4$ by linear interpolation in the following table.

x	0	± 1	± 2	± 3	± 4
$f(x)$	-1	2	1	4	3

We will approximate f by three linear polynomials, i. e., $m = 3$, $n_1 = n_2 = n_3 = 1$, $d_1 = d_2 = d_3 = 0$. One solution is $u = (-4, -2, 0, 4)$, $g_1(x) = -x$, $g_2(x) = -x$, $g_3(x) = x$, $\mu = 1$. Another solution is $v = (-4, 0, 2, 4)$, $g_1(x) = -x$, $g_2(x) = x$, $g_3(x) = x$, $\mu = 1$. The mid-point between u and v, namely $(-4, -1, 1, 4)$, does not provide a solution however, since the best approximator on $[-1,1]$ is $g_2(x) \equiv .5$ and this permits a maximum error of 1.5.

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